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The Theory of Grossone and the Continuum Problem

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1 The Continuum Problem

The Continuum Problem is the problem of whether there exists one intermediate cardinality between $\aleph_0$ and $\mathfrak{c}$.[1] As is known, the problem is unsolvable by the 'standard' axioms of set theory, ZFC. In recent years, new axioms have been investigated which would, purportedly, be able to yield a solution to the problem by extending ZFC. However, there is no broad consensus both on the justifiability of such axioms, and on the appropriateness of the solutions (cf. [Maddy, 1997], [Hamkins, 2015]). Hence, one likely scenario is that the Continuum Problem will keep being seen as unsolved (even 'unsolvable') by most set-theorists.

The (philosophical) question of why it should be so is also a very old question, and it would seem that no univocal answer may be provided.

Cantor himself seems to have already had, at some point, a glimpse of what the troubles with solving the problem might be, insofar as he realised that one could potentially adopt a richer notion of continuum, such as the one propounded by supporters of the *actual infinitesimals* in Cantor's time, then adopted by several other non-standard theories, in the context of which the Continuum Problem has a completely different meaning.[2]

The purpose of the present paper is to approach the problem from an equally radical, and a lot less explored, point of view, according to which:

[1] Cantor's hypothesis (the Continuum Hypothesis) was that there was no such cardinality, ie., that $\mathfrak{c} = \aleph_1$.

[2] For Cantor's own misgivings, emerged in correspondence with Mittag-Leffler, cf. [Dauben, 1979], p. 131, and footnote 37, p. 332. Theories of the non-standard (non-archimedean) continuum are discussed in the historically accurate and exhaustive [Ehrlich, 2006], [Ehrlich, 2012]. A more recent, but equivalent, approach, that of the 'Euclidean continuum', has been introduced in [Benci and Forti, 2020].

1) the concept of 'continuum' does not instantiate a uniquely determinate reality; 2) different 'continua' are observable according to what representation of the real numbers is chosen. This is the point of view of the theory of *grossone* (①), a novel approach to the infinite formulated by Y. Sergeyev in recent years.[3]

As we shall show, if one assumes the correctness of the grossone-theoretic standpoint, the Continuum Problem dissolves or merely reduces to a problem of 'representation' of the real numbers.

2 Grossone and the Continuum

The theory of grossone revolves around the introduction of one single (infinite) quantity, ①. More specifically, [Sergeyev, 2017]'s Infinite Unit Axiom introduces ① as the number of elements of the set $\mathbb{N}$. This is performed by extrapolating from the finite to the infinite the idea that n is both the number of elements of the set $\{1, 2, 3, \cdots, n-1, n\}$ and the last element of this set. Since ① is the last element of the set of natural numbers, one obtains the 'extended' set of the natural numbers:

$$\underbrace{1, 2, ..., ① - 2, ① - 1, ①}_{\text{the set of natural numbers}}, ① + 1, ① + 2, ...$$

It should be noted straight away that, in opposition to Cantor's ω, ① is not an ordinal, and does not denote any property of sets (it isn't a *set*, for that matter). In opposition to non-standard infinite numbers, and contrary to what has been suggested sometimes,[4] ① isn't a non-archimedean quantity obeying the axioms of *real-closed* fields.

Now, through using ①, one can measure infinite sets. But in order to produce such measures, one should first find a suitable representation of the infinite set one is up to measuring. For some sets, the 'representation' is clearly *unique*. For instance, $\mathbb{Z}$ is the set:

$$..., -3, -2, -1, 0, +1, +2, +3, ...$$

[3]The most up-to-date exposition of the theory is in [Sergeyev, 2010] and [Sergeyev, 2017]. Further details on the manifold applications of the theory may be found in the more recent [Sergeyev and De Leone, 2022]. For axiomatic treatments of the theory, see [Lolli, 2015], [Montagna et al., 2015].

[4]For instance, [Benci and Freguglia, 2019] mistakenly assume that ① is comparable to the α of Benci and Di Nasso's α-theory, itself a new version of non-standard analysis, cf. [Benci and Di Nasso, 2003].

so its number of elements is equal to $2① + 1$ (0 is not taken by Sergeyev to be a natural number from the beginning).

In the case of the continuum, that is, of the set of reals $\mathbb{R}$, things are more complicated. By noticing that a real number is just an integer followed by ① instances of 0 or 1, one gets $2^{①}$ reals, the grossone-theoretic measure of the infinite set of all possible sequences of 0 and 1 of length ①.

If, on the other hand, one represents real numbers using, say, three digits, 0, 1, 2, then the measure of *this* continuum will be $3^{①}$. If one ends up using the decimal notation involving ten digits, $0, 1, 2, \ldots, 9$, then one will obtain $10^{①}$ as a measure of this other continuum. Thus, as anticipated, measuring the continuum crucially depends on how one represents ('observes') the continuum itself, a fact which, as is clear, is not captured by the set-theoretic axioms.

As in physics, where one observes the same object by different instruments (e.g., microscope or binoculars) different things are observable in dependence of the chosen instrument. At the moment when the observer has chosen their instrument of observation they have chosen what will be visible.

3 Philosophical Implications

The conception of the continuum (better, 'continua') adopted by the theory of grossone broadly belongs to the class of conceptions which view the continuum as *indeterminate*, at least until one has not chosen the 'tool' one wishes to use to observe it.

This conception has historically been part of constructivists' philosophical agenda. However, constructivists generally deny reality to the continuum insofar as the latter entails: 1) the representability (even existence) of 'actually infinite' objects or, alternatively, 2) the existence of objects for which no effective (constructive) process may be available.[5]

On the contrary, the grossone-theoretic approach neither denies reality to the actual infinite, nor does it reject non-constructive processes of counting.

As far as the former issue is concerned, the theory of grossone admits of actually infinite sets, but is concerned with measuring these sets through

[5] The conception, originally in [Weyl, 1918], has been, once again, expressed, in more recent times, by Feferman, who has claimed that the Continuum Problem is 'inherently vague'. Cf. [Feferman, 1987], and [Feferman, 2014].

using tools 'concretely available' for their observation.[6] As far as the latter issue is concerned, the theory of grossone does not refrain from 'infinite measures', only aims to tie them to the counting process, by thriving on noticeable and persistent analogies between the finite and the infinite.[7]

The grossone-theoretic philosophy of the continuum is, thus, one for which the notions of 'observation', 'observational tool' and 'process of counting' have the utmost relevance.

In the paper, we shall indicate how all these philosophical aspects of the theory of grossone are both variously connected to old foundational viewpoints, such as constructivism or Hilbertian finitism, and, most crucially, to more recent themes arising in the philosophy of mathematical practice (or what is called philosophy of 'real mathematics'), where a strong emphasis is placed on the cognitive aspects of 'counting in the infinite', on anti-foundationalism, on the importance of applications, and on a quasi-observational approach to mathematics based on analogies with the methodology of (physical) sciences.[8]

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[6]This fact is enshrined in [Sergeyev, 2017]'s **Postulate 2.** *Mathematical objects (and their nature) are irrelevant to counting purposes*.

[7]Cf. [Sergeyev, 2017]'s **Postulate 1** as well as **Postulate 3**; the latter is none other than Euclid's Part-Whole Principle: 'the whole is greater than its parts'.

[8]For an overview of all such themes, cf. [Mancosu, 2012].

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