What is a multiverse?

A general framework for the set theoretic multiverse

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1 Introduction

Ten years ago, Hamkins (2012) changed the landscape of the foundations of mathematics, by introducing a novel conception that tried to clarify some ambiguous notions in current set theoretic practice. In particular, he provided a revolutionary interpretation for the practice of forcing: a multiverse of different set theoretic universes. Such an idea immediately sparked an intense debate in the philosophy of set theory and the foundations of mathematics. In the following years, several crucial contributions were made by Carolin Antos (2018), Bagaria and Claudio Ternullo (2020), S. Friedman (2012), Gitman and Hamkins (2011), Koellner (2009), Maddy (2017), Meadows (2021), Martin (2001), Steel (2014), C. Ternullo (2019), Väänänen (2014), Venturi (2020), and Woodin (2011), just to name a few. These contributions can be roughly divided in two broad categories:

- the general debate between *universism* (the position that there is a single, determined universe of sets) and *pluralism* (the position that there are several universes of sets, all of them equally interesting, i.e. the multiverse);[1]
- the introduction of novel mathematical characterisations of the set theoretic multiverse.[2]

Indeed, while the general idea behind pluralism in the philosophy of mathematics is more or less the same every time, the actual mathematical details can vary enormously from one characterisation to the other. We have multiverses based upon different kinds of forcing[3] , multiverses with different background logics[4], multiverses that try to accommodate the highest number of different universes[5], etc. Even though all these different set theoretic multiverses share the same, general, philosophical idea, they

[1]Examples of such papers are Koellner (2013), Martin (2001) and Maddy (2017).

[2]For example, Gitman and Hamkins (2011), Steel (2014) and C. Ternullo and S.-D. Friedman (2016).

[3]For example, Steel (2014) is based upon set-generic forcing, while Venturi (2020) on Robinson infinite forcing.

[4]Väänänen (2014) and S. Friedman (2012) are the prime examples.

[5]Hamkins (2012) is the maximal multiverse conception, encompassing all possible universes.

differ wildly from the mathematical perspective. There are some proposal of assessing all these differences[6], but this research field is still in its infancy.

One of the goals of this research is categorising all the different multiverses, trying to draw distinctions between them and maybe defining some broad categories or types of multiverses. This has been done in several, informal ways in the literature: for example, C. Antos et al. (2015) distinguishes between realist and anti-realist multiverses, appealing to known definition in the ontology of mathematics.

Another possible distinction can be drawn between multiverses in which the truth value of set theoretic statements collapses to the truth value of that same statement in a particular universe of the multiverse *vs* multiverse that don't collapse in this way. Or between *linear* and *branching* multiverses.[7] The linear multiverses expand by building on all the universes part of the multiverse, while the branching ones admits "bifurcations". According to this informal distinction, Steel's set generic multiverse is a linear multiverse, while Hamkins' multiverse is a branching one.

The problem with most of the distinctions found in the literature and folklore on the multiverse is that they are exclusively *philosophically* motivated. While they are very useful in investigating matters in the foundations and philosophy of mathematics, we still lack a purely mathematical characterisation of set theoretic multiverses *as a whole*.

In this paper, I plan to close this gap, and develop a mathematical method to investigate the set-theoretic multiverses as a single, uniform structure. To do so, I interpret the set-theoretic multiverses to be a method to "carve" the collection of all models of set theories in sub-collections that share some common features. I contend that this changes the landscape of the research in the set theoretic multiverse in an important way. While currently each multiverse is investigated singularly, as an isolated entity, with my proposal it will become possible to approach the class of all multiverses as a single, unified structure. As an analogy, each single multiverse can be thought of as an algebra, or a logic, while my approach is similar to Universal Algebra, or Universal Logic.[8] Moreover, I will discuss the main philosophical consequences of this approach for the debate between universism (the position that claims that there exists only one, *unique* set-theoretic universe) and multiversism (the position that instead claims the existence of several, equally legitimate, set-theoretic universes).

2 The Multiverse Operator

Let $\mathcal{M}$ the collection of all models of ZFC, i.e. $\mathcal{M} = Mod(ZFC) = \{\, M \mid M \models ZFC \,\}$. All these models are just chaotically in $\mathcal{M}$, but we can define a relation to try to put some order between them. Consider for example the following relation $R := R(M,N) \iff N = M[G]$, where $M, N \in \mathcal{M}$ and $M[G]$ is a set-generic extension of M. Essentially, this R relates two models of ZFC iff one is the set-generic

[6] See for example Meadows (2022).

[7] A recent paper that brings up this distinction, in the context of potentialist systems, is Hamkins and Linnebo (2022).

[8] See for example Beziau (2007).

extension of the other. We can now define the *closure* of R by taking all the N such that N is a set-generic extension of M, and all the M such that M is a ground of N (i.e. the model on which we apply set-generic forcing to get the extension N). So, we are adding to R *all* the pairs (M, x), where x is a set-generic extension of M, *and all* the pairs (y, N), where y is a ground of N. Taking this closure defines an equivalence relation R^* from R, and partitions the collection of all models of ZFC, $\mathcal{M}$ in equivalence classes. Let M be the representative of any such equivalence class. All the other members of that equivalence class are either a set-generic extension of M, or M is a set-generic extension of them. Consequently, we can think of the set-generic multiverse as the equivalence relation that partitions the collection of all models of ZFC $\mathcal{M}$ in the equivalence classes $[M]_R = \{ x \in \mathcal{M} \mid R(M, x) \vee R(x, M) \}$.

By taking a different relation, it is possible to partition the collection $\mathcal{M}$ in several different ways. For example, we can consider a relation C such that $C(M, N)$ iff N is a class-generic extension of M, or a top-extension, and so on. When taking the closure of all these relations, we are then defining just as many equivalence relations that partitions $\mathcal{M}$ in different ways. Each one of these partitions can then be interpreted as a set-theoretic multiverse (e.g. a class-generic multiverse, if we take C to be relating class-generic extensions). However, changing the relation is not the only source of variation. Another possibility is to change the background collection of models $\mathcal{M}$, for example by considering not only the models of ZFC, but the collection of all the models of ZF, or even weaker theories. The ultimate goal is to investigate the behaviour of all these different relations against the background of the broadest possible collection of models: $\mathcal{V} = Mod(T) = \{ M \mid M \models T \}$, where T is *any* set-theoretic axiomatization. As we will see in what follows, this is the crucial step to go beyond the debate between universism and multiversism. Finally, further variation can be achieved by changing how we take the closure of the basic relation R. The closure described above is a very strong closure, but it is also possible to consider some weaker closures that takes only *some* of the x such that $R(M, x)$. Or, we can consider closing R only "upward" (so only under extensions) or only "downward" (so only under grounds). Each of these weaker closures don't define an equivalence class on $\mathcal{M}$, since there will be some missing N that is not picked up by R.

One particularly interesting question worth highlighting is the kind of properties that the single equivalence classes have. Consider the base case again: $\mathcal{M}$ is the collection of all the models of ZFC, and R^* is the strong closure of the relation R defined on set-generic extensions. Now consider two models $M, N \in \mathcal{M}$, that are not immediately related by R. Can we move from M to N by following R? In this basic case, we already know that we can. To do so, we just need to go upward, by going to an extension G of M following R, and then downward, going to another ground U of G, until we encounter N. Moreover, it doesn't matter whether we go downward or upward a number of times, followed by just as many trips in the opposites relation. We can go upward and then downward, or downward and then upward, but we always end up in the right place. This is true because, for any two models M, N, we can relate them to the same set-generic extension, and also to the same ground, as proved by Hamkins with his set-theoretic geology methods (Fuchs, Hamkins, and Reitz, 2015):

Theorem 1 (Hamkins). *Let M, N be two models of ZFC. Then, there exists a set-generic extension of W such that both M and N are grounds of W. Moreover, there exists a ground G such that both M, N are set-generic extensions of G.*

While this is a very nice property of the equivalence classes defined in the base case, this cannot be generalised to any case. For example, if we consider only the *weak* closure of R, where we take only *some* of the set-generic extensions of M, and only some of the grounds of M, then clearly the theorem fails. This is because we may be missing the common ground between two models, or their common extension. And with the failure of the theorem, we cannot be sure that we can always go from any two models M, N, if not directly connected by R, since a connection between two middle points might be missing.

The last point I want to explore is the definition of different types of multiverses using different relations R. We already saw that we can define generic multiverses in this way, in any of their variants (so set-generic multiverses, class-generic multiverses and hyperclass multiverses). We can also define multiverses based upon different kind of extensions, like top-extensions or end-extensions. But what about other kind of multiverses, for example Friedman's Hyperuniverse (HP)? From the literature, we know that HP is the collection of all countable, transitive models of ZFC, definable from a ground model using the infinitary V-logic. Without getting in too many details, this simple description lets us see a possible method to define multiverses using a relation R on a collection of models: the key lies in adding the right conditions to the definition of R. For example, in the case of HP, the relation between two countable, transitive models of ZFC, M and N, is that one is set-generic extension of the other, *and* it is definable from the ground using V-logic. Consequently, the relation $H := H(M, N) \iff N = M[G]$ *and* N is definable in M using V-logic is a good candidate to define the relation between models inside HP. Taking the closure of this relation H then partitions the collection $\mathcal{M}$ in equivalance classes, each one a hyperuniverse. However, we said that HP is the collection of all *countable, transitive* models of ZFC, so we must change also the background collection of models. Taking $\mathcal{M}^C = \{\, M | M \models ZFC \,\}$, with M countable and transitive will give us exactly the collection needed.

Interestingly, looking at the case of HP gives us also insights on how changing the background collection of models changes the possible ways we can partition it. Consider again the basic case of $\mathcal{M}$, the collection of all models of ZFC, with no assumption on their countability. If we then try to partition this collection by taking the closure of the same P define for HP above, we end up with "Hyperuniverse-like" multiverses. These multiverses are V-logic Multiverses, a new type of multiverse developed by the present author and Claudio Ternullo (Ceglie and C. Ternullo, n.d.), that foregoes the assumption that we are working only with countable models, and instead admits also *uncountable* ones. Consequently, changing the background collection of models has deep consequences on the types of partitionings that I can perform (that is, on the kind of multiverses I can define).

The philosophical upshot of the approach just described is that it changes one of the core assumptions behind the universism *vs* multiversism debate. Recall that, according to the universist, *there exists* only one, unique set-theoretic universe. According to the multiversist, *there exist* several, equally legitimate, universes of set theory. This opposition is purely on platonic grounds: both in the universist and in the multiversist case the debate is about the existence of a particular mathematical object. But such a dichotomy cannot be resolved: it is a purely metaphysical issue that will always be mathematically problematic. The current project instead proposes to change the narrative of this debate. The need of such change has already been argued by Ternullo (C. Ternullo, 2022). His proposal is that we should differentiate between $\mathcal{V}$-models, i.e. the models of any set-theory, that "look like" the universe of set theory, and the trascendent V, that instead cannot be mathematically investigated. My proposal is similar, and highlights a possible way out of the universism-multiversism impasse. To reiterate, according to my approach a set-theoretic multiverse is *not* an actual, mathematical (platonic) object, but just a mathematical method, on the same level as forcing or inner models, that helps grouping models of set theory by some interesting, common feature. The universist's "unique" universe is not a particular model of set theory (a particular $\mathcal{V}$-model, following Ternullo's terminology), but the collection of all possible models of set theory (that can then be interpreted as the "trascendent" V from Ternullo's proposal). The set-theoretic multiverses are a method to put order in this chaotic collection, by grouping some of the models under some unifying relations and features.

3 Concluding remarks

In conclusion, in this paper I proposed a unified framework for the set theoretic multiverse. To do so, I defined a binary relation R on a collection of models of set theory, that maps two models iff they are one the (set-generic, class-generic, etc.)-extension of the other. This opens up the possibility to study the set-theoretic multiverse in a more uniform and unified way, instead that trying to assess each single multiverse by itself.

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